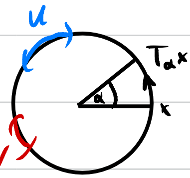
Metric Spaces and Topology
Lecture 17



Example. Irrational rotations are generically ergodic.

Proof. Let T be an irrational rotation, i.e. $T = T_\alpha$ for some $\alpha/2\pi \notin \mathbb{Q}$. Suppose towards a contradiction that $\exists$ Baire meas. E_T-invariant sets B, B^c s.t. neither is meagre. By the 100% lemma, $\exists$ nonempty open sets U, V s.t. U is 100% B and V is 100% B^c. Since U and V are ctbl unions of open intervals/segments, we may assume WLOG that U & V are open intervals. Because $\exists$ dense orbit (in fact all orbits), we can rotate U enough times $n \in \mathbb{Z}$, so that $T^n(U) \cap V \neq \emptyset$. Since $T(B) = B$ and T is homeomorphism (maps meagre to meagre & open to open), it follows that $T^n(U)$ is still 100% B. Thus, the nonempty open set $T^n(U) \cap V$ is both 100% B and 100% B^c, contradicting that S^1 is a Baire space. $\square$

Def. For a graph G on a vertex set X, a (proper) colouring of G is a map $c : X \to Y$, for some set Y, such that adjacent vertices in G receive different c-values (called colours).  When X is a Polish space, a **Borel (resp. Baire meas.) proper colouring** of G is a Borel (resp. Baire meas.) map $c: X \to Y$, for some Polish space Y (e.g. $Y := \{0,1,2\}$ with discrete metric) that is a proper colouring. When Y is ctbl, this is equivalent to each colour being Borel (resp. Baire meas.).

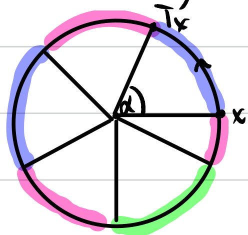
<u>Recall</u> (AC). A graph G can be coloured with 2 colours if and only if G is bipartite ($\Leftrightarrow$ doesn't have odd cycles).

Proof. $\Rightarrow$. An odd cycle cannot be coloured with 2 colours.

$\Leftarrow$ (AC). We choose a point x_C in each component C and colour vertices in C blue if their distance from x_C is even and red otherwise. Because there are no odd cycles, there won't be adjacent vertices whose distance from x_C has the same parity. $\square$

<u>Example</u>. G_{T_α} for an irrational rotation T_α admits a Borel

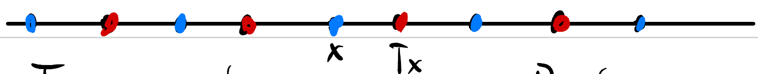
colouring with 3 colours.

Proof. 

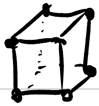
Each colour is a finite union of half-open intervals. □

Corollary (from gen. ergodicity). There is no Baire meas. (in particular, no Borel) colouring of the irrational rotation graph with 2 colours.

Proof. Suppose that is such a colouring, i.e. there are Baire meas. sets B, B^c s.t. $T_\alpha(B) = B^c$.



Since T_α is a homeomorphism, B is meagre $\Leftrightarrow$ B^c is meagre, hence both are nonmeagre because S^1 is a Baire space. But both B and B^c are T_α^2-invariant and $T_\alpha^2 = T_{2\alpha}$, which is still an irrational rotation, hence gen. ergodic, a contradiction. □

Other graphs: Hamming graph and G_0. We define a graph H_0 on $2^{\mathbb{N}}$, called the Hamming graph, as follows: put an edge between $x, y \in 2^{\mathbb{N}}$ if x and y differ by exactly one bit (i.e. index).  is the Hamming graph on 2^3.

Prop. Hamming graph is bipartite ($\iff$ doesn't have odd cycles).

Proof. The coordinate-wise binary sum over a cycle has to be the all-0 sequence 0^∞, thus, it must have even number of summands. In other words, each flipped bit has to be flipped back. $\square$

Thus, using AC, we can colour the Hamming graph H_∞ on $2^{\mathbb{N}}$ can be coloured with 2 colours.

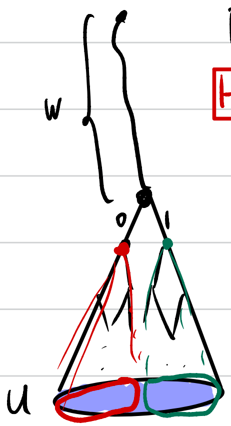
Obs. The connectedness equiv. rel. for the Hamming graph H_∞ on $2^{\mathbb{N}}$ is eventual equality, which is denoted by $\mathbb{E}_0$, i.e. $x \, \mathbb{E}_0 \, y \iff \forall^\infty n \; x(n) = y(n)$.

Each $\mathbb{E}_0$-class is ctbl, thus there are continuum many $\mathbb{E}_0$-classes, i.e. H_∞-components.

Theorem. The Hamming graph H_∞ doesn't have a Baire meas.

colouring with ctbly many colours.

Proof. Suppose on the contrary that there is a Baire meas. colouring $c: 2^{\mathbb{N}} \to \mathbb{N}$, in other words each colour $c^{-1}(n)$ is Baire meas. Since $2^{\mathbb{N}} = \bigsqcup_{n \in \mathbb{N}} c^{-1}(n)$, one of these colours $c^{-1}(n)$ has to be comeagre. By the 100% lemma, there is a nonempty open set U that is 100% that colour $B := c^{-1}(n)$ (say blue). Since U is a disjoint union of cylinders, we may assume that U itself is a cylinder, i.e. $U = [w]$ for some finite word $w \in 2^{<\mathbb{N}}$.



But $U = [w0] \sqcup [w1]$.

$\boxed{\text{HW}}$ The set of neighbours of a meagre set is still meagre. (Changing one bit is a homeomorphism).

The map $f_k : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ which flips the k^{th} bit, $k := \text{length}(w)$, is a homeomorphism, and it maps $[w0]$ onto $[w1]$.

f_k has to map $B \cap [w0]$ onto $B^c \cap [w1]$, which is a contradiction since the former is nonmeagre while the latter is meagre. □

One can define an acyclic graph G_0, a subgraph of the Hamming graph H_∞, which has the same connected components, so each G_0-component is a spanning tree of an H_∞-component. Fix a dense set $\{s_k : k \in \mathbb{N}\} \subseteq 2^{<\mathbb{N}}$ of finite sequences such that $\text{length}(s_k) = k$. (This exists by a homework exercise.)

G_0 is defined as follows: x and y in $2^{\mathbb{N}}$ are adjacent if $\exists k$ s.t. $y = s_k{}^\frown b^\frown z$ and $x = s_k{}^\frown \bar{b}^\frown z$, where $b \in \{0,1\}$ and $\bar{b} := 1-b$.

Fact. The G_0-connectedness relation is still E_0.

Proof. $\boxed{\text{HW}}$

Fact 2. G_0 is acyclic.

Proof. $\boxed{\text{HW}}$

Almost the same proof as for Hamming graph shows:

Theorem. G_0 doesn't have a Baire meas. colouring with ctbly many colours.

Note that if G is a graph on a Polish space X and $\exists$ Borel graph homomorphism $h: 2^{\mathbb{N}} \to X$ from G_0 to G, then G also doesn't have a Baire meas. ctbl colouring (by composition).

G_0-dichotomy (Kechris - Solecki - Todorcevic). Let G be a Borel graph on a Polish space X (i.e. the edges of G form a Borel subset of X^2).

Then:

either: G has a Borel ctbl colouring

or else: $\exists$ continuous graph-homomorphism $h: 2^{\mathbb{N}} \to X$ from G_0 to G.